Chapter 2

The Monte-Carlo Method and applications to option pricing

2.1 The Monte-Carlo method

The basic principle of the Monte-Carlo method is to implement on a computer the strong law of large numbers (SLLN): if $(X_k)_{k\geq 1}$ is a sequence, defined on a prob. space $(\Omega, \mathcal{A}, \mathbb{P})$, of independent random variables with the same distribution as X, then:

$$\mathbb{P}(d\omega)\text{-a.s.} \quad \overline{X}_M = \frac{X_1(\omega) + \dots + X_M(\omega)}{M} \xrightarrow[M\to\infty]{} m_X := \mathbb{E}(X)$$

The sequence (X_k) is also called an i.i.d. sequence of random variables with distribution $\mu = \mathbb{P}_X$.

Conditions to implement this MC method on a computer:

* generate pseudo-random numbers $(U_k)_{k\geq 1}$ of law $\mathcal{U}([0,1])$
* generate X as a function of $U_1, U_2, \dots$:
  $X = \varphi_\tau(U_1, \dots, U_\tau)$ (τ is a finite stopping rule)

2.1.1. Rates of convergence

The (weak) rate of convergence in the SLLN is governed by the Central Limit Theorem (CLT) which says that if $X \in L^2(\mathbb{R})$ then:

$$\sqrt{M}\,(\overline{X}_M - m_X) \xrightarrow[M\to\infty]{law} \mathcal{N}(0; \sigma_X^2)$$

where $\sigma_X^2 = \mathrm{Var}(X)$

the mean quadratic error is exactly:

$$\| \overline{X}_M - m_x \|_2 = \mathbb{E}\left((\overline{X}_M - m_x)^2 \right)^{1/2}$$
$$= \frac{\sigma_x}{\sqrt{M}}$$

(indeed: $\mathbb{E}\left((\overline{X}_M - m_x)^2 \right) = \mathbb{E}\left(\left(\sum_{i=1}^{M} \frac{X_i - m_x}{M} \right)^2 \right)$

$$= \sum_{i=1}^{M} \frac{\mathbb{E}\left((X_i - m_x)^2 \right)}{M^2}$$

$$= \frac{M \sigma_x^2}{M^2} = \frac{\sigma_x^2}{M}$$)

This error is also known as the RMSE (Root Mean Square Error)

2.1.2. Confidence level and confidence interval

The CLT also reads: $\sqrt{M}\left(\frac{\overline{X}_M - m_x}{\sigma_x} \right) \xrightarrow[(M \to \infty)]{\text{loi}} \mathcal{N}(0;1)$

so for all a, b in $\mathbb{R}$ ($a < b$):

$$\lim_{M \to \infty} \mathbb{P}\left(\sqrt{M}\left(\frac{\overline{X}_M - m_x}{\sigma_x} \right) \in [a;b] \right) = \Phi(b) - \Phi(a)$$

where Φ is the cumulative distribution function of $\mathcal{N}(0,1)$

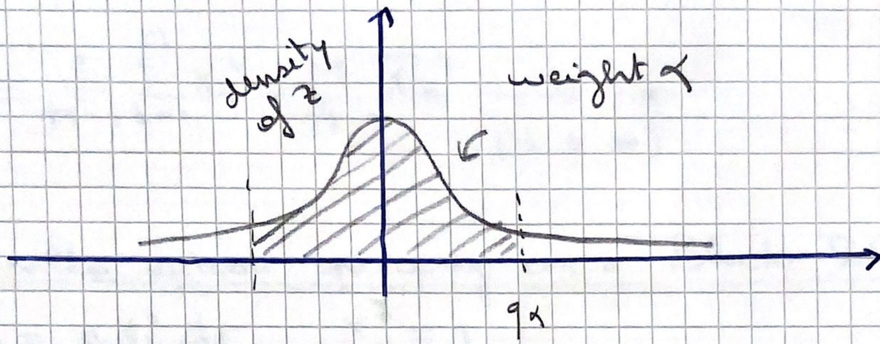
$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$$

We usually assume $\sqrt{M}\left(\frac{\overline{X}_M - m_x}{\sigma_x} \right)$ is of law $\mathcal{N}(0,1)$.

Let $\alpha \in (0,1)$ denote a confidence level (close to 1) and let q_α be the two-sided quantile defined as the unique sol. to:

$$\mathbb{P}(|Z| \leq q_\alpha) = \alpha \qquad \text{i.e.} \qquad 2\Phi(q_\alpha) - 1 = \alpha$$

$(Z \sim \mathcal{N}(0,1))$



Small computation: we want $\mathbb{P}(|Z| \leq q_\alpha) = \alpha$,

This is: $\quad \mathbb{P}(Z > q_\alpha) + \mathbb{P}(Z < q_\alpha) = 1-\alpha$

these two are equal by symmetry

$$2\,\mathbb{P}(Z > q_\alpha) = 1-\alpha$$
$$2(1-\Phi(q_\alpha)) = 1-\alpha$$
$$2\Phi(q_\alpha) - 1 = \alpha$$
$$\Phi(q_\alpha) = \frac{1+\alpha}{2}$$

We take $a = q_\alpha$ and $b = -q_\alpha$ and we can define a theoretical confidence interval at level α:

$$J_M^\alpha = \left[\overline{X}_M - q_\alpha \times \frac{\sigma_X}{\sqrt{M}},\ \overline{X}_M + q_\alpha \frac{\sigma_X}{\sqrt{M}}\right]$$

which satisfies: $\mathbb{P}(m_X \in J_M^\alpha)$

$$\overset{\shortparallel}{\mathbb{P}}\left(\sqrt{M}\left|\frac{\overline{X}_M - m_X}{\sigma_X}\right| \leq q_\alpha\right)$$

$$\xrightarrow[(M\to\infty)]{} \mathbb{P}(|Z| \leq q_\alpha) = \alpha$$

However: J_M^α is <u>theoretical</u> because we do not know σ_X.

We usually replace σ_X^2 by its Monte-Carlo estimate:

$$V_M = \frac{1}{M-1}\sum_{k=1}^{M}(X_k - \overline{X}_M)^2 = \frac{1}{M-1}\sum_{k=1}^{M} X_k^2 + \frac{M}{M-1}\overline{X}_M^2$$
$$- \frac{2\overline{X}_M}{M-1}\sum_{k=1}^{M} X_k$$

$$\dots = \frac{1}{M-1}\sum_{i=1}^{M} X_i^2 - \frac{M}{M-1}\overline{X}_M^2 \xrightarrow[(M\to\infty)]{} \sigma_X^2$$

2.1.3 Vanilla option pricing in a Black-Scholes model

$$\begin{cases} dX_t^0 = r X_t^0\,dt, & X_0^0 = 1 \\ dX_t^1 = X_t^1(r\,dt + \sigma_1 dW_t^1), & X_0^1 = x_0^1 \\ dX_t^2 = X_t^2(r\,dt + \sigma_2 dW_t^2), & X_0^2 = x_0^2 \end{cases}$$

$W = (W^1, W^2)$ is a bi-dim. Brownian motion such that $\langle W^1, W^2\rangle_t = \rho t$ $\quad (\rho \in [-1,1])$.

In other words, we can decompose
$W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2}\,\widetilde{W}_t^2$ where $(W^1, \widetilde{W}^2)$ is a standard 2-dim. Brownian motion.

filtration: $(\mathcal{F}_t)_{t\in[0,T]} = \sigma(W_s,\ 0\le s\le t,\ \mathcal{N}_{\mathbb{P}})$

$\mathcal{N}_{\mathbb{P}}$: augmented with $\mathbb{P}$-negligible sets

$\mathbb{P}$ is the risk-neutral probability, meaning that
$e^{-rt}X_t = e^{-rt}\begin{pmatrix} X_t^0 \\ X_t^1 \\ X_t^2\end{pmatrix}$ is a $(\mathbb{P}, (\mathcal{F}_t))$-martingale

For all t: $X_t^0 = e^{rt}$, $X_t^i = x_0^i \exp\left[\left(r - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t^i\right]$ $\quad (i = 1, 2)$

(this is an application of Itô's Lemma)

A European vanilla option with maturity $T>0$ is an option related to a European payoff:

$$h_T = h(X_T)$$

which only depends on X at time T. In such a complete market, the option premium a time $t\in[0,T]$ is given by $V_t = e^{-r(T-t)}\,\mathbb{E}(h(X_T)\,|\,\mathcal{F}_t)$

$$\left(V_0 = e^{-rT}\,\mathbb{E}(h(X_T))\right)$$

As W has independent stationary increments, we have:
$$\left(\frac{X_T^i}{X_t^i}\right)_{i=1,2} \overset{law}{=} \left(\frac{X_{T-t}^i}{x_0^i}\right)_{i=1,2}.$$

We define $v(x_0, T) = e^{-rt}\,\mathbb{E}(h(X_T))$ and we compute
$$\begin{aligned}
V_t &= e^{-r(T-t)}\,\mathbb{E}(h(X_T)\mid\mathcal{F}_t) \\
&= e^{-r(T-t)}\,\mathbb{E}\left(h\left[\left(X_t^i \times \frac{X_T^i}{X_t^i}\right)_{i=1,2}\right]\mid\mathcal{F}_t\right) \\
&= e^{-r(T-t)}\left[\mathbb{E}\,h\left(\left(x^i \times \frac{X_{T-t}^i}{x_0^i}\right)_{i=1,2}\right)\right]_{x^i = X_t^i,\, i=1,2} \\
&= v(X_t, T-t)
\end{aligned}$$

Examples: * Vanilla call with strike price K:
$$h(x^1, x^2) = (x^1 - K)_+$$
There is a closed form (for V_T): Black–Scholes formula.

* Best-of-call with strike price K: $h_T = (\max(X_T^1, X_T^2) - K)_+$

↳ a quasi-closed form exists but we will price it using a Monte-Carlo simulation

We re-write: $e^{-rT} h_T \overset{law}{=} \varphi(Z^1, Z^2)$
$$:= \Big[\max\Big(x_0^1 \exp\Big(-\frac{\sigma_1^2}{2}T + \sigma_1\sqrt{T}\,Z^1\Big),\; x_0^2 \exp\Big(-\frac{\sigma_2^2}{2}T + \sigma_2\sqrt{T}\big(\rho Z^1 + \sqrt{1-\rho^2}\,Z^2\big)\Big)\Big) - Ke^{-rt}\Big]_+$$

where $Z = (Z^1, Z^2) \sim \mathcal{N}(0, I_2)$

We take a M-sample $(Z_m)_{1\le m\le M}$ and get
$$(\text{Best-of-Call}) = \mathbb{E}(\varphi(Z^1, Z^2)) \simeq \varphi_M := \frac{1}{M}\sum_{m=1}^{M}\varphi(Z_m)$$

Estimate for the variance:

$$\overline{V}_M(\varphi) = \frac{1}{M-1}\sum_{m=1}^{M} \varphi(Z_m)^2 - \frac{M}{M-1}\overline{\varphi}_M^{\,2} \simeq \mathrm{Var}(\varphi(Z))$$

(for M large enough).

Level of confidence: $\alpha \in (0,1)$.

Confidence interval:

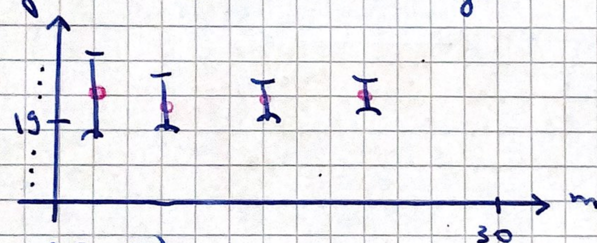
$$I_M^{\alpha} = \left[\overline{\varphi}_M - q_\alpha\sqrt{\frac{\overline{V}_M(\varphi)}{M}}\ ,\ \overline{\varphi}_M + q_\alpha\sqrt{\frac{\overline{V}_M(\varphi)}{M}}\right]$$

where q_α is defined by: $2\Phi(q_\alpha) - 1 = \alpha$.

<u>Numerical application:</u>

$r = 0.1$, $\sigma_i = 0.2 = 20\%$, $\rho = 0.5$, $X_0^i = 100$, $T=1$, $K=100$

Idea of what one should get



$M = 2^n$ $(n = 1, \dots, 30)$

($\alpha = 95\%$)

<u>Methodology</u>

Monte-Carlo simulation to compute $m_X = \mathbb{E}(X)$

* specification of a confidence interval $\alpha \in (0,1)$
* simulation of a M sample, computation of empirical mean $\overline{X}_M$ and variance $\overline{V}_M$
* computation of the resulting confidence interval I_M at confidence level α